

# Directed percolation with an absorbing boundary

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## Abstract

We consider directed percolation with an absorbing boundary in  $1 + 1$  and  $2 + 1$  dimensions. The distribution of cluster lifetimes and sizes depend on the boundary. The new scaling exponents can be related to the exponents characterizing standard directed percolation in  $1 + 1$  dimension. In addition, we investigate the backbone cluster and red bonds, and calculate the distribution of living sites along the absorbing boundary.

*Key words:* Directed percolation; chemical reactions; spatio-temporal intermittency

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## 1 Introduction

Directed percolation (DP) is the common name for spreading processes with active (“live”) sites and an absorbing (“dead”) state. The process of directed percolation is more precisely defined on a lattice where at each time step a site can become alive with probability  $p$ , if and only if at least one of its neighbors was alive at the previous time step [1]. For  $p \rightarrow 0$  the process rapidly terminates whereas for  $p \rightarrow 1$  the live sites spread without limit. There exists

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a critical value  $p = p_c$  such that the distribution of clusters originating from an initial live site is a power law. The probability that there are living sites at time  $t$  after the introduction of one living site in a infinite lattice of dead sites at time  $t = 0$  is [2]

$$P(t) \sim t^{-\delta}, \quad \delta = \beta/\nu_{\parallel}. \quad (1)$$

Here  $\beta$  is the order parameter exponent which determines the probability  $\epsilon^\beta$  ( $\epsilon \equiv p - p_c$ ) for the cluster to survive to infinity, and  $\nu_{\parallel}$  and  $\nu_{\perp}$  are the correlation length exponents along ( $\xi_{\parallel} \sim \epsilon^{-\nu_{\parallel}}$ ) and perpendicular ( $\xi_{\perp} \sim \epsilon^{-\nu_{\perp}}$ ) to the time direction of the DP process. Both  $\beta$  and  $\nu_{\parallel}$  enter in Eq. (1) which reflect the fact that the final survival of the cluster is determined when the cluster size reaches the correlation length  $\xi_{\parallel}$ ; the probability to reach  $\xi_{\parallel}$  scales as  $\epsilon^\beta \sim \xi_{\parallel}^{-\beta/\nu_{\parallel}}$  in accordance with Eq. (1).

Directed percolation is known to be equivalent to Reggeon field theory [3,4] which describes the evolution of a density  $\rho$  of live sites for a large class of processes. An understanding of facets of directed percolation have implications for the understanding of a number of widely different problems in physics, including the dynamics of chemical reactions and catalyzers [1,2], contact processes [5], spatio-temporal intermittency [6–8], self-organized criticality [9–12], and directed polymers [13].

In the present paper we consider directed percolation with an absorbing boundary [14,15]. In the context of spatio-temporal intermittency [6–8], the process of directed percolation can be viewed as lateral convection of intermittent spots along a boundary which is absorbing (i.e., the boundary layer has laminar flow). Thus the active sites represent intermittent spots that originate from one point on the boundary due to a impurity or an edge. The decay of the turbulent spots into the absorbing (laminar flow) could possibly be connected to directed percolation with a wall. Another more traditional picture is a catalytic process that is initiated at the edge of the catalyzer. As a result, the propagation of active live sites can then only propagate into the system. We consider very slow activation on the boundary: the activity in the system is supposed to be entirely determined by the propagation of one initial spark at the boundary at time zero.

We consider time-directed percolation in a geometry as shown in figure 1, and will refer to the  $d$  transverse direction as space ( $x$ ) and to the longitudinal direction as time ( $t$ ). The boundary absorbs the parts of the DP cluster which in an open geometry would have continued spreading on the other side. We will first discuss the  $1 + 1$ -dimensional ( $d = 1$ ) directed site DP on a square lattice where  $p_c = 0.705485$ ,  $\beta = 0.2765$ ,  $\nu_{\parallel} = 1.7338$ , and  $\nu_{\perp} = 1.0968$  [15–18]; in the following, the subscript '1' denotes critical exponents for DP with an absorbing boundary. Recently, series expansions for  $1 + 1$ -dimensional bond

directed percolation in a similar absorbing geometry were carried out [15]. Our results in  $d = 1$  are in complete agreement with the estimates in [15].

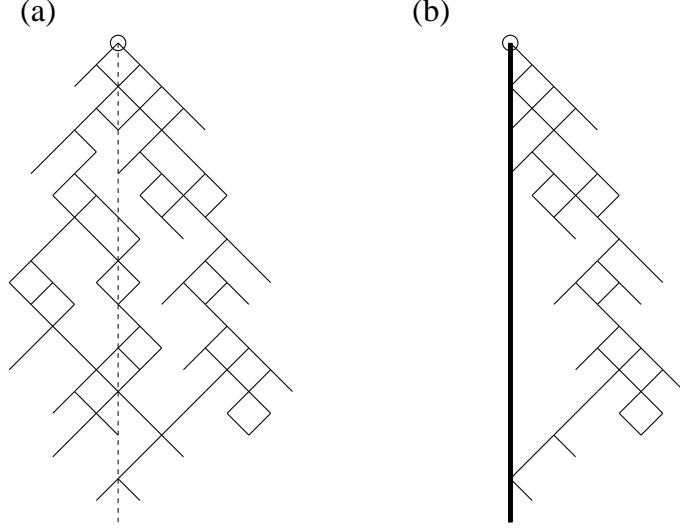


Fig. 1. a) Schematic drawing of a DP cluster. b) Schematic drawing of the same DP cluster but now with the absorbing boundary present. One notices the existence of a different distribution of voids compared to the case (a). The time axis is directed downwards. Note that in our implementation of the absorbing boundary we allow for activity on the boundary; not allowing for this simply amounts to shifting the absorbing boundary one lattice spacing to the left.

## 2 Critical behavior

First we notice that there remain a critical value  $p_c$  for DP with an absorbing boundary, and that this value is independent of the presence of such a boundary. In figure 2 we display the meandering (squared)  $x^2(t)$  and the mass  $m(t)$  of the cluster measured for a value of  $p$  close to  $p_c$  ( $\epsilon = -0.0015$ ). The meandering scales as  $\xi_{\perp}^2 \sim t^{2\nu_{\perp}/\nu_{\parallel}}$ , with  $t$  determined by  $\xi_{\parallel}$ . The mass scales as  $\xi_{\parallel}\xi_{\perp}\epsilon^{\beta} \sim t^{1+(\nu_{\perp}-\beta)/\nu_{\parallel}}$ . We find that both these quantities scale in the same way as if there was no boundary. Thus, both the meandering exponent  $\chi \equiv \nu_{\perp}/\nu_{\parallel} = 0.6327$  and the mass scaling exponent  $1 + (\nu_{\perp} - \beta)/\nu_{\parallel} = 1.4732$  are independent of the absorbing boundary. We also measure the lifetime distribution and average lifetime (see below) and conclude that the three exponents ( $\beta$ ,  $\nu_{\parallel}$ , and  $\nu_{\perp}$ ) for DP are still present for DP with the presence of a wall. This is consistent with the fact that DP clusters with a wall consist of subsets of DP clusters without a wall. Therefore the correlation length and thus the correlation length exponents should be identical in the two cases. Further, for the subset of clusters that survive to a given time  $t$ , there exist a

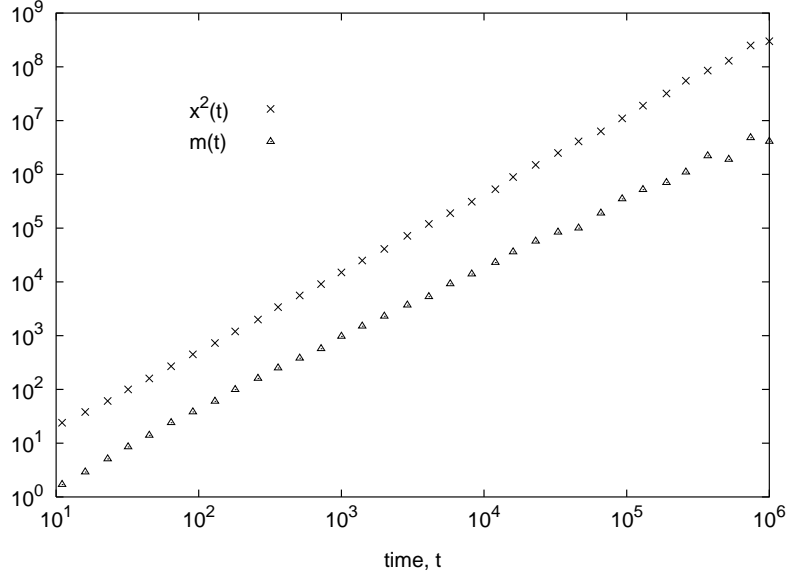


Fig. 2. Cluster mass,  $m(t)$ , and meandering,  $x^2(t)$ , as function of time in  $1 + 1$  dimensions. We obtain the mass scaling exponent  $1 + (\nu_{\perp} - \beta)/\nu_{\parallel} = 1.475 \pm 0.005$ , and the meandering exponent  $2\chi \equiv 2\nu_{\perp}/\nu_{\parallel} = 1.265 \pm 0.005$ , in agreement with the values for DP without an absorbing boundary.

connected leftmost path closest to the wall that merge the starting point with a point that survive to time  $t$ . To the right of this path the cluster is identical to a normal DP cluster with the same scaling behavior of the density. The fact that the scaling of the mass of a DP cluster with and without a wall are identical show that for the subset of clusters that survive to a given time, the active points within these clusters are on average so far from the wall that the density is not influenced by the presence of the wall.

Next, we investigate the distribution of lifetimes in the case of an absorbing boundary. In figure 3 we show the probability that there are still living sites at time  $t$ . We find that the presence of an absorbing boundary changes the exponent of the lifetime distribution from the one given by (1) (i.e.,  $\delta = 0.1594$ ) to

$$P_1(t) \sim t^{-\delta_1}, \quad \delta_1 = 0.420 \pm 0.005. \quad (2)$$

We have  $\delta_1 = \beta_1/\nu_{\parallel}$ , where the order parameter exponent  $\beta_1$  determines the probability  $\epsilon^{\beta_1}$  for the cluster to survive to infinity. This yields the value  $\beta_1 = 0.728 \pm 0.008$ , and we conclude that  $\beta_1 \neq \beta$ . Note, as mentioned, that  $\beta$  is still needed for DP with a boundary in order to describe the scaling of the cluster mass near  $p = p_c$ . From the full scaling behavior

$$P_1(t) = t^{-\delta_1} f(t\epsilon^{\nu_{\parallel}}), \quad (3)$$

we obtain the average lifetime

$$\langle t \rangle \sim \epsilon^{-\tau_1}, \quad \tau_1 = \nu_{\parallel}(1 - \delta_1) = \nu_{\parallel} - \beta_1. \quad (4)$$

In Ref. [15] they obtain  $\tau_1 = 1.000\dots$  and conjecture that  $\tau_1$  is exactly equal to unity. This leads to  $\beta_1 = \nu_{\parallel} - 1$ , and  $\delta_1 = 1 - 1/\nu_{\parallel} = 0.4232$  in agreement with our result. We also measure  $\langle t \rangle$  directly (Fig. 4) and obtain the estimate  $\tau_1 = 1.000 \pm 0.005$ , yielding an estimate for  $\beta_1$  in agreement with the one above. Even though we do not know of an analytical argument for  $\tau_1 = 1$ , we can obtain an upper bound by the following heuristic argument: if we assume that the statistics of the DP cluster are described by the statistics of the “outer envelope” which scales as  $t^{\chi}$  we obtain that the first return of the envelope back to the absorbing boundary determines the cluster lifetime distribution. The probability that the envelope returns to “zero” (i.e., to the absorbing boundary) is accordingly given by the scaling of the first return of a fractional Brownian motion with exponent  $\chi$ :

$$p'(t) = \frac{1}{t^{2-\chi}}. \quad (5)$$

This yields the survival exponent  $\delta' = (2 - \chi) - 1 = 1 - \nu_{\perp}/\nu_{\parallel}$ , and thus the estimate  $\tau_1' = \nu_{\perp}$ , only a little larger than the correct value. The reason that the ‘envelope’ argument is only approximate is due to the fact that clusters can spontaneously die out before the envelope returns to the boundary.

Our results for the probability  $P_1(s)$  to have a cluster of size larger than  $s$  show that it follows a power law with an exponent  $\tau_s - 1 = 0.283 \pm 0.005$ . This value can easily be explained in terms of the lifetime distribution investigated above. A cluster of lifetime  $t$  will have a size (mass)  $s$  of the order  $s \sim t^{1+(d\nu_{\perp}-\beta)/\nu_{\parallel}}$ , where the density exponent  $\beta$  enters (and not the order parameter exponent  $\beta_1$  associated with a seed on the boundary). Transforming the probability distribution (3) for  $P_1(t)$ , we thus obtain

$$P_1(s) = s^{-(\tau_s-1)} g(s\epsilon^{\nu_{\parallel}+d\nu_{\perp}-\beta}), \quad \tau_s = 1 + \frac{\beta_1}{\nu_{\parallel} + d\nu_{\perp} - \beta}. \quad (6)$$

This yields  $\tau_s = 1.287$  in nice agreement with our numerical result in  $1 + 1$  dimension. Compared to the value  $1 + \beta/(\nu_{\parallel} + \nu_{\perp} - \beta) = 1.108$  for DP with no boundary we find as expected that the boundary suppresses the cluster sizes. From Eq. (6) we find the average cluster size

$$\langle s \rangle \sim \epsilon^{-\gamma_1}, \quad \gamma_1 = \nu_{\parallel} + d\nu_{\perp} - \beta - \beta_1, \quad (7)$$

which replaces the usual hyperscaling relation when there is no wall present.

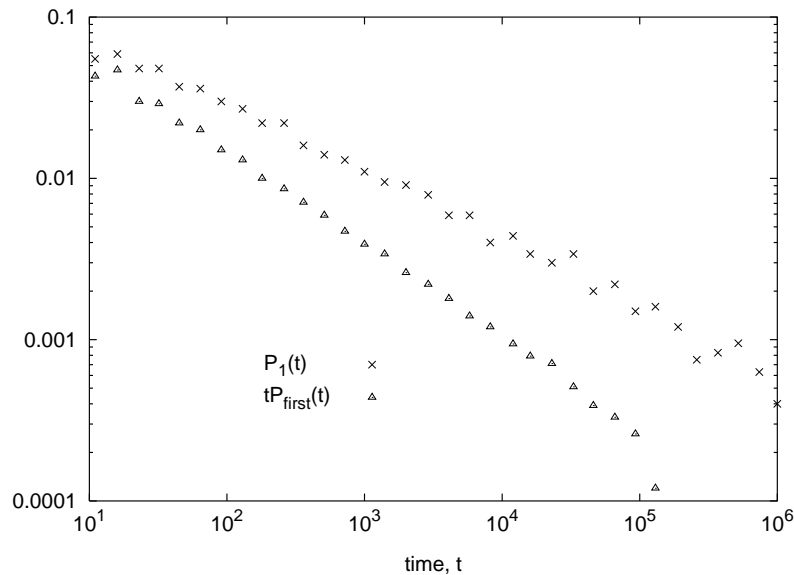


Fig. 3. Probability  $P_1(t)$  that there are still live sites at time  $t$  in  $1 + 1$  dimensions. We obtain a power law with exponent  $\delta_1 = 0.42 \pm 0.01$ . We also display the distribution  $p_{\text{first}}(t)$  of first return along the absorbing boundary and obtain a power-law distributions with exponent  $\tau_{\text{first}} = 1.577 \pm 0.005$ .

In standard DP the cluster mass and the order parameter scale with the same exponent and there are three independent exponents. In DP with an absorbing boundary, we find that the mass and the order parameter scale with different exponents. One of these exponents ( $\beta_1$ ) can however be related to the other exponents by assuming that  $\tau_1 = 1$  and we are left with the same three independent exponents as for standard DP in  $1 + 1$  dimensions.

### 3 Boundary activity and backbone

Now consider the activity on the absorbing boundary. Direct measurement (by box counting) of the fractal dimension of living sites along the time direction yields a dimension  $D_1 = 0.578 \pm 0.005$ . This is an accordance with the result in figure 3 of an independent measurement of the distribution of time intervals in which the boundary is dead, i.e., the distribution of first return  $p_{\text{first}}(t)$  of activity. From figure 3 we obtain  $p_{\text{first}}(t) \sim 1/t^{\tau_{\text{first}}}$  with  $\tau_{\text{first}} = 1.577 \pm 0.005$ , confirming the scaling formula  $\tau_{\text{first}} = 1 + D_1$ . We also obtain that the distribution of all return  $p_{\text{all}}(t) \sim 1/t^{\tau_{\text{all}}}$  fulfills  $\tau_{\text{all}} = 1 - D_1$ . Without an absorbing boundary, the dimension of the activity along an arbitrarily longitudinal cut of the DP cluster would be  $D_{\parallel} = 1 - \beta/\nu_{\parallel} = 0.841$ . Thus, the activity on a cut is much less when DP can evolve only on one side of this cut than when the activity can move freely back and forth between both sides.

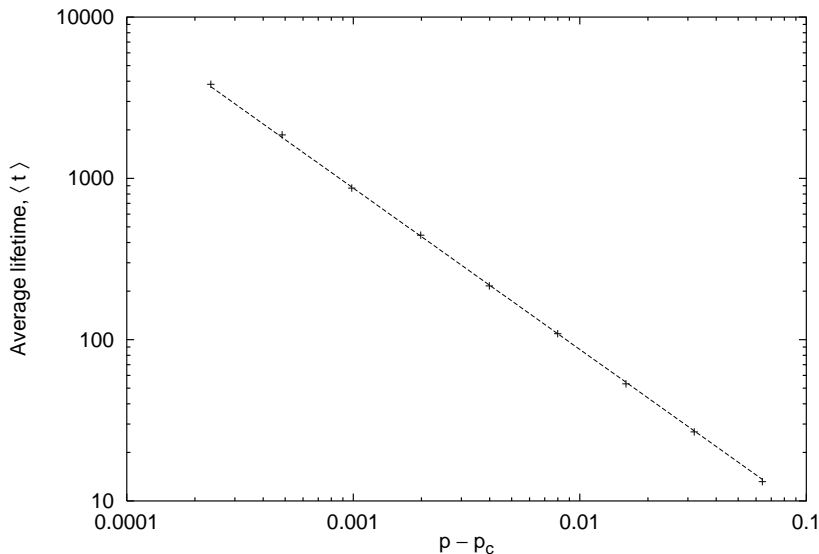


Fig. 4. Average lifetime as function of  $p$  for values above  $p_c$  in  $1 + 1$  dimensions. The estimate for the slope is  $\tau_1 = 1.000 \pm 0.005$ .

The connection between DP with a wall and the field-theoretic formulation [14] has been worked out in Ref. [19]. Therein it is shown that  $\beta_1$  determines the scaling of the density  $\epsilon^{\beta_1}$  on the wall. Thus, with the boundary we obtain the relation  $D_1 = 1 - \beta_1/\nu_{\parallel} = 1/\nu_{\parallel}$  in nice agreement with our numerical results.

The backbone is obtained from the infinite cluster by removing all dangling ends. Thus the backbone consists of precisely those bonds which would be occupied by both the time-directed DP process and its reversed time-directed process. It then follows that the backbone density  $\epsilon^{\beta^{BB}}$  is described by the exponent  $\beta^{BB} = 2\beta$ . We have numerically confirmed that this backbone behavior is still valid for DP with a wall. Besides, we have measured the backbone dimension on the wall with the result  $D_1^{BB} = 1 - \beta_1^{BB}/\nu_{\parallel} = 0.16 \pm 0.01$ . This yields  $\beta_1^{BB} = 1.46$  in nice agreement with  $\beta_1^{BB} = 2\beta_1$ , cf. the result for the bulk.

On the backbone one can identify the so-called red bonds [20] which, if one is cut, divide the cluster into two parts. A renormalization group argument [21] (see also [11]) yields for the number of red bonds up to scale  $t$  the scaling

$$N_R(t) \sim t^{1/\nu_{\parallel}}. \quad (8)$$

We have measured the scaling of red bonds for DP with a wall and obtained results in complete agreement with Eq. (8). In addition we have measured

the scaling of the red bonds along a longitudinal cut for DP with no wall and obtained the result  $N_R^{cut}(t) \sim t^{-0.04 \pm 0.02}$ . This is in accordance with the expected behavior  $N_R^{cut}(t) \sim t^{1/\nu_{\parallel} - \chi} \sim t^{-0.056}$ , where the extra factor originates from the scaling of the width  $\xi_{\perp}$  of the cluster. Finally, we have measured the scaling of red bonds on the wall for DP with the wall and found the behavior  $N_{R,1}(t) \sim t^{-0.60 \pm 0.1}$ .

## 4 Results in 2 + 1 dimensions and concluding remarks

The above procedure can be applied to higher dimensions. For 2 + 1 dimensional bond directed percolation on a bcc lattice we have  $p_c = 0.287338$ ,  $\beta = 0.584$ ,  $\nu_{\parallel} = 1.295$ ,  $\nu_{\perp} = 0.734$ ,  $\delta = 0.451$ , and  $\chi = \nu_{\parallel}/\nu_{\perp} = 0.567$  [22,23]. We have measured the lifetime distribution, cluster distribution, and average lifetime and cluster distributions. Again, we conclude that  $\nu_{\parallel}$  and  $\nu_{\perp}$  are unchanged. Furthermore,  $\beta$  is needed in order to describe the scaling of the mass whereas a new exponent,  $\beta_1$ , is needed to describe the lifetime distribution: we find the values  $\beta_1 = 1.05 \pm 0.05$ , and  $\tau_1 = 0.26 \pm 0.02$  in accordance with  $\tau_1 = \nu_{\parallel} - \beta_1$ , cf. Eq. (1). In contrast to the 1 + 1 dimensional case, in 2 + 1 dimensions  $\beta_1$  does not seem to be (simply) related to the other exponents and we conclude that there are four independent exponents. For 2 + 1 dimensional systems with an edge (which introduces yet another exponent) see Ref. [19].

One may speculate what the effect of the boundary will be if the cluster is initiated some fixed distance  $x$  away from the boundary. In that case, scaling will be dominated by the DP behavior (1) until the meandering of the cluster allows it to reach the absorbing boundary, i.e., until a crossover time  $t_{\times} \propto x^{\nu_{\parallel}/\nu_{\perp}}$ . After this crossover time, the distribution of cluster survival times will become steeper, and be determined by (2). Thus the presence of a boundary will cause two scaling regimes for the cluster size. One can also consider the case where the DP process is initiated in between two absorbing boundaries. In such a case the DP process cannot evolve infinitely. After some time, of the order of  $L^{\nu_{\parallel}/\nu_{\perp}}$ , where  $L$  is the distance between the boundaries, the otherwise critical DP process dies out.

In summary, we have investigated the effects of an absorbing boundary on DP and discussed the novel scaling behavior originating from such a boundary. The scaling can be understood in terms of the three exponents for DP plus one new exponent. Our results are in agreement with recent series expansion estimates. We also study the scaling at the boundary. We stress that boundary effects are not limited to the case where the boundary is absorbing—it could, e.g., also be reflecting. The effect that reduces the lifetimes of the clusters and the fractal dimension of activity on the boundary is solely due to the fact that living branches cannot meander back and forth over the boundary. Finally, the



exponent which determines the lifetime distribution in 1+1 dimensions is  $1/\nu_{\parallel}$ , which describes the scaling of the red bonds. In the context of self-organized interface growth determined by DP paths there exists a relation which involves  $1/\nu_{\parallel}$  and dictates that the corresponding average ‘lifetime’ should scale as (4) with an exponent equal to unity [10]. It would be interesting to explore these similarities and to possibly relate the lifetime distribution to the scaling of the red bonds.

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## References

- [1] F. Schögl, Z. Phys. 253 (1972) 147.
- [2] P. Grassberger and A. De la Torre, Ann. Phys. 122 (1979) 373.
- [3] P. Grassberger and K. Sundermeyer, Phys. Lett. B 77 (1978) 220.
- [4] J. L. Cardy and R. L. Sugar, J. Phys. A 13 (1980) L423.
- [5] T. M. Liggert, *Interacting Particle Systems* (Berlin, Springer Verlag, 1985).
- [6] Y. Pomeau, Physica D 23 (1986) 3.
- [7] H. Chaté and P. Manneville, Phys. Rev. Lett. 58 (1987) 112.
- [8] J. Rolf, T. Bohr and M. H. Jensen, unpublished.
- [9] L.-H. Tang and H. Leschhorn, Phys. Rev. Lett. 70 (1993) 3833.
- [10] M. Paczuski, S. Maslov and P. Bak, Phys. Rev. E 53 (1996) 414.
- [11] G. Huber, M. H. Jensen and K. Sneppen, Phys. Rev. E 52 (1995) R2133.
- [12] M. Markošová, M. H. Jensen, K. B. Lauritsen and K. Sneppen, Phys. Rev. E 55 (1997) R2085.
- [13] N. I. Lebedev and Y.-C. Zhang, J. Phys. A 28 (1995) L1.
- [14] H. K. Janssen, B. Schaub and B. Schmittmann, Z. Phys. B 72 (1988) 111.

- [15] J. W. Essam, A. J. Guttmann, I. Jensen and D. TanlaKishani, J. Phys. A 29 (1996) 1619.
- [16] J. W. Essam, K. De'Bell, J. Adler and F. M. Bhatti, Phys. Rev. B 33 (1986) 1982.
- [17] J. W. Essam, A. J. Guttmann and K. De'Bell, J. Phys. A 21 (1988) 3815.
- [18] I. Jensen, J. Phys. A 29 (1996) 7013.
- [19] P. A. Fröjdh, M. J. Howard and K. B. Lauritsen, in preparation (1997).
- [20] H. E. Stanley, J. Phys. A 10 (1977) L211; R. Pike and H. E. Stanley, *ibid.* 14 (1981) L169.
- [21] A. Cognilio, Phys. Rev. Lett. 46 (1981) 250; J. Phys A 15 (1982) 3824.
- [22] P. Grassberger, J. Phys. A 22 (1989) 3673.
- [23] P. Grassberger and Y.-C. Zhang, Physica A 224 (1996) 169.